L1

September 3rd, 2025

1.1 Lipschitz Condition

Lipschitz Condition Consider the function $f(t, x)$ with $f : \mathbb{R}^{n+1} \to \mathbb{R}^n, |t - t_0| \leq a, x \in D \subset \mathbb{R}^n$. f satisfies the Lipschitz condition with respect to x if in $[t_0 - a, t_0 + a|] \times D$ we have:

$$||f(t, x_1) - f(t, x_2)|| \leq L||x_1 - x_2|| \tag{1.1}$$

with $x_1, x_2 \in D$ and L a constant. L is called the Lipschitz constant.

If f satisfies the condition, f i Lipschitz continuous in x. This also implies continuity in x.

1.1 Consider the initial value problem:

$$\dot{x} = f(x, t), \quad x(t_0) = x_0 \tag{1.2}$$

with $x \in D \subset \mathbb{R}^n, |t - t_0| \leq a; D = x|||x - x_0|| \leq d$, a and d are positive constants. The vector function $f(x, t)$ satisfies the following conditions:

1. $f(x, t)$ is continuous in $G = [t_0 - a, t_0 + a] \times D$
2. $f(x, t)$ is Lipschitz continuous in x

Then, the initial value problem has one and only one solution for $|t - t_0| \leq \min(a, \frac{d}{M})$, with $M = \sup_G ||f||$.

1.2 Gronwall's Inequality

1.2 Assume that for $t_0 \leq t \leq t_0 + a$, with a a positive constatn, we have the estimate:

$$\phi(t) \leq \delta_1 \int_{t_0}^{t_1} \psi(s)\phi(s)ds + \delta_3 \tag{1.3}$$

in which, for $t_0 \leq t \leq t_0 + a, \phi(t)$ and $\psi(t)$ are continuous functions, $\phi(t) \geq 0$ and $\psi(t) \geq 0; \delta_1$ and δ_3 are positive constants. Then we have for $t_0 \leq t \leq t_0 + a$:

$$\phi(t) \leq \delta_3 e^{\delta \int_{t_0}^{t}} \psi(s)ds \tag{1.4}$$

1.3 Assume that for $t_0 \leq t \leq t_0 + a$, with a a positive constant, we have the estimate:

$$\phi(t) \leq \delta_2(t - t_0) + \delta_1 \int_{t_0}^{t} \phi(s)ds + \delta_3 \tag{1.5}$$

in which for $t_0 \leq t \leq t_0 + a$, $\phi(t)$ is a continuous function, $\phi(t) \geq 0; \delta_1, \delta_2, \delta_3$ are constants with $\delta_1 > 0, \delta_2 \geq 0, \delta_3 \geq 0$. Then we have for $t_0 \leq t \leq t_0 + a$:

$$\phi(t) \leq \left(\frac{\delta_2}{\delta_1} + \delta_3\right) e^{\delta_1(t - t_0)} - \frac{\delta_2}{\delta_1} \tag{1.6}$$

1.4 Consider the equation $\dot{x} = f(x, t), x \in \mathbb{R}^n, f : \mathbb{R}^{n+1} \to \mathbb{R}^n, t \geq 0; f(t, x)$ satisfies the Lipschitz condition (Lipschitz constant L) and is continuous in t and x. Consider the initial value problems:

$$\dot{x} = f(x, t), x(0) = a, \text{solution} x_0(t) \text{on interval} I \tag{1.7}$$

$$\dot{x} = f(x, t), x(0) = a + \eta, \text{solution} x_\epsilon(t) \text{on interval} I \tag{1.8}$$

If $||\eta|| \leq \epsilon$ (ϵ real, positive) we have:

$$||x_0(t) - x_\epsilon(t)|| \leq \epsilon e^{Lt} \text{on interval} I \tag{1.9}$$

L2 Autonomous Equations (1)

September 10th, 2025

In autonomous equations, the independent variable t does not occur explicitly, e.g. $\dot{x} = f(x)$. A scalar equation of order n is often written as:

$$x^{(n)} + F(x^{(n-1)}, \dots, x) = 0 \tag{2.1}$$

in which $x^{(k)} = d^k x/dt^k, k = 0, 1, \dots, n, x^{(0)} = x$.

In characterizing the solutions of autonomous equations we use three special solution sets:

1. Equilibrium or Stationary solutions
2. Periodic solutions
3. Integral manifolds

2.1 Phase-Space, Orbits

The translation property says that if we have a solution $\phi(t)$ of an autonomous equation in the domain $D \subset \mathbb{R}^n$, then $\phi(t - t_0)$ with t_0 a constant is also a solution. Note that the two are two different solutions, but correspond to the same orbital curves in phase space.

If $x \in D \subset \mathbb{R}^n$, D is called phase-space, and for autonomous equations this is often separately studied.

EXAMPLE Considering the harmonic equation:

$$\ddot{x} + x = 0 \tag{2.2}$$

The equation is autonomous. We can put $x = x_1$ and $\dot{x} = x_2$ to obtain:

$$\dot{x_1} = x_2 \tag{2.3}$$

$$\dot{x_2} = -x_1 \tag{2.4}$$

The solutions are linear combinations of $\cos t$ and $\sin t$. The solution space can be sketched in the $x, \dot{x}$ plane. This is the phase-plane.

END EXAMPLE

The space in which we describe the behavior of the variables x^i, parameterized by t, is called phase-space.

A point in phase-space with coordinates $x_1(t), \dots, x_n(t)$ for a certain t is called a phase-point. In general, for increasing t, a phase-point shall move through phase-space.

In carrying out the projection, we generally do not know the solution curves. A DE can easily be formulated to describe the behavior of the orbits in phase-space. We can write the autonomous equation out in components like:

$$\dot{x_i} = f_i(x), \quad i = 1, \dots, n \tag{2.5}$$

We shall use one of the components of x as a new independent variable. This requires that $f_i(x) \neq 0$. With the chain rule we obtain $(n-1)$ equations:

$$\begin{aligned} \frac{dx_j}{dx_i} &= \frac{f_j(x)}{f_i(x)} \\ \vdots \;&\; \vdots \\ \frac{dx_n}{dx_i} &= \frac{f_n(x)}{f_i(x)} \end{aligned}$$

Solutions of this system in phase-space are called orbits. If existence and uniqueness applies to the autonomous equation, it also applies to the above system. This means orbits will not intersect.

2.2 Critical Points and Linearization

Critical Point The point $x = a$ with $f(a) = 0$ is called a critical point of equation $\dot{x} = f(x)$.

A critical point corresponds with an equilibrium solution of the equation.

Attractor A critical point of the equation is called a positive attractor if there exists a neighborhood $\Omega_a \subset \mathbb{R}^n$ of $x = a$ such that $x(t_0) \in \Omega_a$ implies $\lim_{t\to\infty} x(t) = a$. If for $t \to -\infty$, it is a negative attractor.

In analysis of critical points and equilibrium solutions we always start by linearizing the function using the Taylor series expansion around the critical point.

2.3 Periodic Solutions

Periodic Solution Suppose that $x = \phi(t)$ is a solution of the equation $\dot{x} = f(x), x \in D \subset \mathbb{R}^n$ and suppose there exists a positive number T such that $\phi(t + T) = \phi(t)$ for all $t \in \mathbb{R}$. Then $\phi(t)$ is called a periodic solution of the equation with period T.

Consider phase-space corresponding with an autonomous equation. For a periodic solution we have that after a time T, $x = \phi(t)$ assumes the same value in $\mathbb{R}^n$. Thus, a periodic solution produces a closed orbit or cycle in phase-space.